Dynamics of the Jacobian Matrices Arising in three-dimensional Euler equations: Application of Riccati Theory

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An attempt is made to remove, at least partially, the nonlocal nature associated with the pressure term of the incompressible Euler equations. We consider the dynamics of the Jacobian matrix $\boldsymbol{J}(t)$ relating spatial and material coordinates in incompressible three-dimensional Euler equations. By applying the theory of matrix-valued Riccati equations to the velocity gradient tensor $\boldsymbol{V}(t)$ assuming time-symmetric pressure Hessian, we derive an identity which is local in material coordinates for a fluid particle with $\det \boldsymbol{V}(0) \neq 0$, as a result of invariance under time-reversal. This imposes constraints on the time evolution of the velocity gradients and generates a chain of exact infinite relationships on the Taylor coefficients of $\boldsymbol{V}(t)$ in time. Some of the first few are explicitly given. As a corollary, we prove that if the evolution of the velocity gradient is symmetric w.r.t. time $\boldsymbol{V}(t) = \boldsymbol{V}(-t)$, it must be a constant and $\boldsymbol{J}(t) = \exp(\boldsymbol{V}(0)t)$. The key results herein may be summarised in the equations (3.10) and (3.11).

Keywords: incompressible Euler equations, Riccati equation, Jacobian matrix

1. Introduction

The three-dimensional Euler equations for an incompressible fluid are known to have nonlocal terms stemming from the pressure gradient term:

$$\frac{D\boldsymbol{u}}{Dt} = -\nabla p, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.2}$$

where $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + (\boldsymbol{u} \cdot \nabla)$ is the material derivative. If we use vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ we may eliminate the pressure term in its governing equations to obtain

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u},$$

but the nonlocal character remains in the integral relationship between vorticity and rate of strain essentially through Biot-Savart law.

In this paper we attempt to remove, at least partially, this cumbersome nonlocal nature of the Euler equations. To this end we consider yet another representation, that is, the one in terms of the velocity gradient $\boldsymbol{V} \equiv \nabla \boldsymbol{u}$ ($V_{ij} = \partial_j u_i$). Its dynamical equations are given by

$$\frac{D\boldsymbol{V}}{Dt} = -\boldsymbol{V}\boldsymbol{V} - \boldsymbol{P}, \tag{1.3}$$

and are also affected by the pressure Hessian $\boldsymbol{P} = \nabla\nabla p$. These may be regarded as a matrix version of Riccati equations. For matrix formulations in fluid dynamics, see e.g. Yudovich (2000), Childress (2001), Yakubovich & Zenkovich (2001), Bennet (2006). Regarding various aspects of the role of the pressure Hessian in vortex dynamics, see e.g. Ohkitani & Kishiba (1995), Gibbon *et al.* (1999), Chae (2006), Constantin (2008).

As an example we consider a relationship

$$\frac{D^2\boldsymbol{\omega}}{Dt^2} = -\boldsymbol{P}\cdot\boldsymbol{\omega}, \tag{1.4}$$

which can be derived from the Euler equations or the vorticity equations.

An illustrative way of derivation is given as follows, e.g. Craik (1994). Introducing an auxiliary variable $\boldsymbol{W}$ in the spirit of Riccati equation as

$$\boldsymbol{V} = \frac{D\boldsymbol{W}}{Dt}\boldsymbol{W}^{-1} \tag{1.5}$$

then it follows from (1.3) that

$$\frac{D^2\boldsymbol{W}}{Dt^2} = -\boldsymbol{P}\boldsymbol{W}. \tag{1.6}$$

By writing (1.5) as

$$\frac{D\boldsymbol{W}}{Dt} = \boldsymbol{V}\boldsymbol{W},$$

it is clear that each column vector of $\boldsymbol{W} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \boldsymbol{w}_3]$ is material:

$$\frac{D\boldsymbol{w}_i}{Dt} = \boldsymbol{V}\boldsymbol{w}_i, \quad \text{for } i = 1, 2, 3.$$

We deduce (1.4) by taking one of them, say, $\boldsymbol{w}_1$ as the vorticity vector. This derivation is based upon the observation that (1.3) is a Riccati equation. We will pursue this line further in what follows. In particular, we will study the dynamics of the Jacobian matrix $\boldsymbol{J}$ which satisfy

$$\frac{D\boldsymbol{J}}{Dt} = \boldsymbol{V}\boldsymbol{J},$$

and

$$\frac{D^2\boldsymbol{J}}{Dt^2} = -\boldsymbol{P}\boldsymbol{J}.$$

We note some other works on Riccati equations in fluid mechanics, e.g. Drazin & Reid (1981), Constantin (2000) and Gibbon (2002).

2. Riccati equation

We recapitulate two basic properties of Riccati ordinary differential equations here. Consider an ODE of the form

$$\frac{dy}{dx} = -y^2 - P(x), \quad y(0) = y_0, \tag{2.1}$$

where $P(x)$ is a given function of x. The following two basic properties are well known.

(1) By setting $y = \dfrac{u'}{u}$, we have a linear equation of the second-order

$$u'' = -P(x)u, \tag{2.2}$$

where we have denoted $' = d/dx$. No methods of obtaining general solutions are known. However, if a particular solution, say, $y_1(x)$ is obtained, then we may get general solutions by a quadrature.

(2) In fact, by setting $y = y_1(x) + u(x)$ we find

$$u' + 2y_1(x)u = -u^2. \tag{2.3}$$

By further setting $z = 1/u$ we reduce the problem to

$$z' - 2y_1 z = 1, \tag{2.4}$$

which is an easily solvable linear equation.

Its solution is given by

$$z = e^{2\int_0^x y_1(s)ds}\left(\int_0^x e^{-2\int_0^{x'} y_1(s)ds}dx' + C\right), \tag{2.5}$$

or, in the original variable, we have

$$u(x) = \frac{1}{e^{2\int_0^x y_1(s)ds}\left(\int_0^x e^{-2\int_0^{x'} y_1(s)ds}dx' + \frac{1}{u(0)}\right)}. \tag{2.6}$$

Below we will apply a similar method to three-dimensional Euler equations.

3. Application to three-dimensional Euler equations

(*a*) *Heart of the matter*

In §1 we have already applied the property (1) to the three-dimensional Euler equations. The aim of this paper is to apply the other property (2) to them.

But there is a catch to that. In the case of ODEs, $P(x)$ is an externally given function of x, which has nothing whatsoever to do with initial data and which is not affected by the solution. So, we may choose $u(0)$ at our disposal, and generate infinitely many different solutions. However, in the case of the three-dimensional Euler equations we do not have such a wide range of freedom, because the velocity and the pressure are inherently related with each other. At first glance, it may seem impossible to apply and to make use of the property (2).

However, it is crucial to recall that the three-dimensional Euler equations describe ideal fluids and that they form a conservative system which satisfies invariance under time-reversal. We then realize that we can still apply the property (2), because if there is one solution, then there is another solution which is running backward in time. So, we have two solutions and it is worth relating them by the property (2).

More specifically, consider an equation for the Laplacian of the pressure

$$\triangle p = -\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}.$$

Normally, we use it to find the pressure, assuming the velocity on the right-hand side is known. Now we take a look at it in a reverse direction as

$$\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} = -\triangle p$$

and notice that it is invariant under $\boldsymbol{u} \to -\boldsymbol{u}$. The relationship between the velocity and the pressure is *not* 1 to 1 but 2 to 1. This recognition makes it possible to apply the property (2) of the Riccati theory to three-dimensional Euler equations.

(*b*) *Exact relationship*

If we replace $\boldsymbol{V}$ in (1.3) as $\boldsymbol{V} \to \tilde{\boldsymbol{V}} = \boldsymbol{V} + \boldsymbol{U}$, we have

$$\frac{D\boldsymbol{U}}{Dt} = -\boldsymbol{V}\boldsymbol{U} - \boldsymbol{U}\boldsymbol{V} - \boldsymbol{U}^2.$$

Setting $\boldsymbol{Z} = \boldsymbol{U}^{-1}$ we find

$$\frac{D\boldsymbol{Z}}{Dt} = \boldsymbol{V}\boldsymbol{Z} + \boldsymbol{Z}\boldsymbol{V} + \boldsymbol{I}.$$

This inhomogeneous equation can be solved as follows, see Whyburn (1934), Reid(1946, 1963, 1972) and Williams (1989).

We first consider homogeneous equations

$$\frac{D\boldsymbol{T}}{Dt} = \boldsymbol{V}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{V}.$$

Their solutions are known to have the form

$$\boldsymbol{T} = \boldsymbol{J}\boldsymbol{C}\tilde{\boldsymbol{J}},$$

where $\boldsymbol{J}$ and $\tilde{\boldsymbol{J}}$ satisfy

$$\frac{D\boldsymbol{J}(t)}{Dt} = \boldsymbol{V}(t)\boldsymbol{J}(t) \tag{3.1}$$

and

$$\frac{D\tilde{\boldsymbol{J}(t)}}{Dt} = \tilde{\boldsymbol{J}(t)}\boldsymbol{V}(t). \tag{3.2}$$

Here, the initial conditions are

$$\boldsymbol{J}(0) = \tilde{\boldsymbol{J}}(0) = \boldsymbol{I}.$$

As are other variables, the Jacobian matrix $\boldsymbol{J}(\boldsymbol{a},t)$ is a function of material coordinates $\boldsymbol{a}$ and time t. We suppress $\boldsymbol{a}$ for simplicity in many places in the present paper.

Second, by the method of variations of constants we may absorb the inhomogeneous term by setting

$$\boldsymbol{J}\frac{D\boldsymbol{C}}{Dt}\tilde{\boldsymbol{J}} = \boldsymbol{I},$$

or

$$\frac{D\boldsymbol{C}}{Dt} = \boldsymbol{J}^{-1}\boldsymbol{I}\tilde{\boldsymbol{J}}^{-1}.$$

Thus we get

$$\boldsymbol{C}(t) = \int_0^t \boldsymbol{J}(s)^{-1}\tilde{\boldsymbol{J}}^{-1}(s)ds + \boldsymbol{C}_1,$$

where $\boldsymbol{C}_1$ is a constant and

$$\boldsymbol{Z}(t) = \boldsymbol{J}(t)\left(\int_0^t \boldsymbol{J}(s)^{-1}\tilde{\boldsymbol{J}}(s)^{-1}ds + \boldsymbol{C}_1\right)\tilde{\boldsymbol{J}}(t). \tag{3.3}$$

Fixing $\boldsymbol{C}_1$ by the initial condition, we obtain

$$\tilde{\boldsymbol{V}}(t) = \boldsymbol{V}(t) + \tilde{\boldsymbol{J}}(t)^{-1}\left(\int_0^t \boldsymbol{J}(s)^{-1}\tilde{\boldsymbol{J}}(s)^{-1}ds + \boldsymbol{U}(0)^{-1}\right)^{-1}\boldsymbol{J}(t)^{-1}.$$

This connects a particular solution $\boldsymbol{V}(t)$ with another one $\tilde{\boldsymbol{V}}(t)$, where we have assumed

$$\det \boldsymbol{V}(0) \neq 0.$$

In fact, besides the original one $\boldsymbol{V}(t)$, we have only one more solution which is a time reversal of the original, that is,

$$\tilde{\boldsymbol{V}}(t) = -\boldsymbol{V}(-t).$$

We finally obtain under the assumption that $\boldsymbol{P}(t) = \boldsymbol{P}(-t)$ for a fluid particle

$$-\left(\boldsymbol{V}(t) + \boldsymbol{V}(-t)\right) = \tilde{\boldsymbol{J}}(t)^{-1}\left(\int_0^t \boldsymbol{J}(s)^{-1}\tilde{\boldsymbol{J}}(s)^{-1}ds - (2\boldsymbol{V}(0))^{-1}\right)^{-1}\boldsymbol{J}(t)^{-1} \tag{3.4}$$

as the general relationship relating the Jacobian matrix and the velocity gradient. In what follows we discuss the consequences of the relationship.

(*c*) *Special case*

For the time being, we shall assume that the time evolution of the velocity gradients is symmetric w.r.t. time, that is,

$$\boldsymbol{V}(t) = \boldsymbol{V}(-t) \tag{3.5}$$

holds for a fluid particle.† Now let consider the dynamics of $\tilde{\boldsymbol{J}}(t)$. If we define $\boldsymbol{L}(t) \equiv \boldsymbol{J}^{-1}(t)$, it satisfies

$$\frac{D\boldsymbol{L}(t)}{Dt} = -\boldsymbol{L}(t)\boldsymbol{V}(t).$$

† For example, in the Taylor-Green vortex, the total enstrophy $Q(t) = \frac{1}{2}\int|\boldsymbol{\omega}(\boldsymbol{x},t)|^2 \boldsymbol{dx}$ satisfies $Q(t) = Q(-t)$; see, e.g. Brachet *et al.* (1984). This does not mean that it has the property (3.5) point-wise.

Under the assumption (3.5), $\tilde{\boldsymbol{J}}(t)$ can be written as

$$\tilde{\boldsymbol{J}}(t) \equiv \boldsymbol{L}(-t) = \boldsymbol{J}(-t)^{-1}.$$

Under the same assumption it is clear that both the vorticity $\boldsymbol{\omega}$ and the rate-of-strain $\boldsymbol{S}$ is invariant under $t \to -t$ as is seen in

$$\boldsymbol{\Omega}(t) = \frac{\boldsymbol{V}(t) - \boldsymbol{V}^T(t)}{2}, \quad \boldsymbol{S}(t) = \frac{\boldsymbol{V}(t) + \boldsymbol{V}^T(t)}{2},$$

where $\boldsymbol{\Omega}$ is the vorticity tensor $\Omega_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k$, or equivalently $\omega_i = -\epsilon_{ijk}\Omega_{jk}$.

Since the left-hand side of Cauchy invariant $\boldsymbol{\omega} = \boldsymbol{J}\boldsymbol{\omega}_0$ is even in t, we have a consistency condition

$$\left(\boldsymbol{J}(t) - \boldsymbol{J}(-t)\right)\boldsymbol{\omega}_0 = 0.$$

It should be noted that in general $\boldsymbol{J}(t)$ is not necessarily an even function of time.

By plugging (3.4, 3.5) into (3.1), we find the following equation for the Jacobian matrices

$$\frac{D\boldsymbol{J}}{Dt}(t) = -\frac{1}{2}\boldsymbol{J}(-t)\left[\int_0^t \boldsymbol{J}(s)^{-1}\boldsymbol{J}(-s)ds - (2\boldsymbol{V}(0))^{-1}\right]^{-1}, \tag{3.6}$$

or, equivalently

$$\frac{D\boldsymbol{J}}{Dt}(-t) = \frac{1}{2}\boldsymbol{J}(t)\left[\int_0^t \boldsymbol{J}(-s)^{-1}\boldsymbol{J}(s)ds + (2\boldsymbol{V}(0))^{-1}\right]^{-1}, \tag{3.7}$$

for $t \geq 0$. If we set

$$\boldsymbol{A}(t) \equiv \int_0^t \boldsymbol{J}(s)^{-1}\boldsymbol{J}(-s)ds - (2\boldsymbol{V}(0))^{-1},$$

we find by (3.6)

$$\begin{aligned}
\frac{D}{Dt}\det\boldsymbol{A} &= \det\boldsymbol{A}\,\mathrm{tr}(\dot{\boldsymbol{A}}\boldsymbol{A}^{-1}) = -2\det\boldsymbol{A}\,\mathrm{tr}\left(\boldsymbol{J}^{-1}(t)\boldsymbol{J}(-t)\boldsymbol{J}^{-1}(-t)\frac{D\boldsymbol{J}}{Dt}\right)\\
&= -2\det\boldsymbol{A}\,\mathrm{tr}\left(\boldsymbol{J}^{-1}(t)\frac{D\boldsymbol{J}}{Dt}\right) = -2\det\boldsymbol{A}\,\mathrm{tr}\left(\frac{D\boldsymbol{J}}{Dt}\boldsymbol{J}^{-1}(t)\right)\\
&= -2\det\boldsymbol{A}\,\mathrm{tr}(\boldsymbol{V}(t)) = 0.
\end{aligned}$$

Hence the matrix in (3.6) is always non-singular. This means if $\boldsymbol{J}(t)$ blows up, some of the elements of $\boldsymbol{A}$ must do so. (In fact, this does not happen under (3.5) as we see below.) To summarise, under the conditions $\boldsymbol{V}(t) = \boldsymbol{V}(-t)$, $\det\boldsymbol{V}(0) \neq 0$, and $\left(\boldsymbol{J}(t) - \boldsymbol{J}(-t)\right)\boldsymbol{\omega}_0 = 0$, we obtain (3.6).

(*d*) *Solution of* $\boldsymbol{J}(t)$ *in the special case*

In fact, it turns out that we may solve (3.6) exactly. By assuming that $\boldsymbol{J}(t)$ has a Taylor expansion in t, we find by successive differentiation

$$\dot{\boldsymbol{J}}(0) = \boldsymbol{V}(0), \ddot{\boldsymbol{J}}(0) = \boldsymbol{V}(0)^2, \dddot{\boldsymbol{J}}(0) = \boldsymbol{V}(0)^3, \ldots$$

and we find

$$\boldsymbol{J}^{(n)}(0) = \boldsymbol{V}(0)^n, \quad \text{for} \quad n \geq 0$$

by induction. We thus find a simple solution

$$\begin{aligned}\boldsymbol{J}(t) &= \boldsymbol{I} + t\boldsymbol{V}(0) + \frac{t^2}{2}\boldsymbol{V}(0)^2 + \frac{t^3}{3!}\boldsymbol{V}(0)^3 + \dots \\ &= \exp(\boldsymbol{V}(0)t). \end{aligned} \tag{3.8}$$

This result can be checked by a direct substitution into (3.6). † It has a local expression in that it depends only on quantities associated with $\boldsymbol{a}$. It is of interest to compare to it the general case, where nonlocality enters at the second-order in time series

$$\boldsymbol{J}(t) = \boldsymbol{I} + t\boldsymbol{V} - \frac{t^2}{2}\boldsymbol{P} + O(t^3).$$

Under the assumption (3.5), we deduce

$$\boldsymbol{V}(t) \equiv \boldsymbol{V}(0),$$

that is, the velocity gradient is a constant. Plugging (3.8) into

$$\boldsymbol{\omega}(t) = \boldsymbol{J}(t)\boldsymbol{\omega}(0)$$

this is possible, only if we have $\boldsymbol{V}(0)\boldsymbol{\omega}(0) = 0$, or

$$\boldsymbol{S}(0)\boldsymbol{\omega}(0) = 0,$$

which means that there is no vortex-stretching at $t = 0$.

(*e*) *General case*

Now we release the condition of (3.5) and see what we get in the more general case. In this case, we have by (3.4)

$$-\tilde{\boldsymbol{J}}(t)\left(\boldsymbol{V}(t) + \boldsymbol{V}(-t)\right)\boldsymbol{J}(t) = \left(\int_0^t \boldsymbol{J}(s)^{-1}\tilde{\boldsymbol{J}}(s)^{-1}ds - (2\boldsymbol{V}(0))^{-1}\right)^{-1}. \tag{3.9}$$

By defining a matrix $\boldsymbol{\Phi}(t) \equiv \tilde{\boldsymbol{J}}(t)\boldsymbol{J}(t)$, we find by direct computations

$$\frac{D\boldsymbol{\Phi}}{Dt} = 2\tilde{\boldsymbol{J}}(t)\boldsymbol{V}(t)\boldsymbol{J}(t). \tag{3.10}$$

The identity (3.9) may then be recast in a neat form

$$\frac{1}{2}\frac{D\boldsymbol{\Phi}(t)}{Dt} + \tilde{\boldsymbol{J}}(t)\boldsymbol{V}(-t)\boldsymbol{J}(t) = -\left(\int_0^t \boldsymbol{\Phi}(s)^{-1}ds - (2\boldsymbol{V}(0))^{-1}\right)^{-1}. \tag{3.11}$$

For the special case $\boldsymbol{V}(t) = \boldsymbol{V}(-t)$ we obtain a closed equation for $\boldsymbol{\Phi}(t)(\equiv \boldsymbol{J}(-t)^{-1}\boldsymbol{J}(t),$ in this case)

$$\frac{D\boldsymbol{\Phi}}{Dt} = -\left[\int_0^t \boldsymbol{\Phi}(s)^{-1}ds - (2\boldsymbol{V}(0))^{-1}\right]^{-1}. \tag{3.12}$$

† It is possible that actually $\boldsymbol{V}(0) = 0$ is the only possibility. We do not discuss this here.

It is readily confirmed that it has a solution of the form

$$\boldsymbol{\Phi}(t) = \exp\left(2\boldsymbol{V}(0)t\right),$$

as it should.

It should be noted that the set of equations (3.10) and (3.11) do not form a closed system; the second term on the left-hand side of (3.11) is similar to the right-hand side of (3.10) but differs by a negative sign in $\boldsymbol{V}(-t)$. Nevertheless, the identity (3.9) tells something about the time evolution of the Euler equations. It generates a chain of infinite exact relationships on the Taylor coefficients of $\boldsymbol{V}(t)$.

Given that

$$\boldsymbol{V}(t) = \sum_{n=0}^{\infty} \boldsymbol{V}_n t^n,$$

we find by (3.1, 3.2)

$$\boldsymbol{J}(t) = \sum_{n=0}^{\infty} \boldsymbol{J}_n t^n, \ \text{ with } \ \boldsymbol{J}_n = \frac{1}{n}\sum_{k=0}^{n-1} \boldsymbol{V}_k \boldsymbol{J}_{n-1-k},$$

and

$$\tilde{\boldsymbol{J}}(t) = \sum_{n=0}^{\infty} \tilde{\boldsymbol{J}}_n t^n, \ \text{ with } \ \tilde{\boldsymbol{J}}_n = \frac{1}{n}\sum_{k=0}^{n-1} \tilde{\boldsymbol{J}}_{n-1-k} \boldsymbol{V}_k.$$

The left-hand side of (3.9) reads

$$-2 \sum_{p,q,r=0}^{\infty} \tilde{\boldsymbol{J}}_p \boldsymbol{V}_{2q} \boldsymbol{J}_r t^{p+2q+r}.$$

The expressions for $\boldsymbol{J}^{-1}$ and $\tilde{\boldsymbol{J}}^{-1}$ read

$$\boldsymbol{J}^{-1}(t) = \sum_{n=0}^{\infty} (\boldsymbol{J}^{-1})_n t^n,$$

where

$$(\boldsymbol{J}^{-1})_0 = \boldsymbol{I}, \ \ (\boldsymbol{J}^{-1})_n = -\sum_{p=1}^{n} \boldsymbol{J}_p (\boldsymbol{J}^{-1})_{n-p}, (n \geq 1),$$

and

$$\tilde{\boldsymbol{J}}^{-1}(t) = \sum_{n=0}^{\infty} (\tilde{\boldsymbol{J}}^{-1})_n t^n,$$

where

$$(\tilde{\boldsymbol{J}}^{-1})_0 = \boldsymbol{I}, \ \ (\tilde{\boldsymbol{J}}^{-1})_n = -\sum_{p=1}^{n} (\tilde{\boldsymbol{J}})_p (\tilde{\boldsymbol{J}}^{-1})_{n-p}, (n \geq 1),$$

respectively.

The bracketed part on the right-hand side of (3.9) reads

$$\sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{p=0}^{n-1} (\boldsymbol{J}^{-1})_p (\tilde{\boldsymbol{J}}^{-1})_{n-1-p} \right) t^n - (2\boldsymbol{V}(0))^{-1} \equiv \sum_{n=0}^{\infty} \boldsymbol{R}_n t^n.$$

By taking its inverse, we finally obtain from (3.9)

$$-2\sum_{p,q,r=0}^{\infty}\tilde{\boldsymbol{J}}_p\boldsymbol{V}_{2q}\boldsymbol{J}_r t^{p+2q+r}=\boldsymbol{R}_0^{-1}-\boldsymbol{R}_0^{-1}\sum_{n=1}^{\infty}\left(\sum_{p=1}^{n}\boldsymbol{R}_p\boldsymbol{R}_{n-p}^{-1}\right)t^n, \tag{3.13}$$

where $\boldsymbol{R}_0^{-1}=-2\boldsymbol{V}(0)$.

We give a few low-order constraints as examples, omitting the details of the straightforward derivations. At $n=1$ we find that both sides are equal to $-4\boldsymbol{V}(0)^2$ consistently. At $n=2$ we find

$$\boldsymbol{V}_1\boldsymbol{V}_0+\boldsymbol{V}_0\boldsymbol{V}_1+2\boldsymbol{V}_2=0,$$

or

$$\boldsymbol{V}_2=-\frac{\boldsymbol{V}_1\boldsymbol{V}_0+\boldsymbol{V}_0\boldsymbol{V}_1}{2}. \tag{3.14}$$

This implies if $\boldsymbol{V}_0$ and $\boldsymbol{V}_1$ are known we can tell what $\boldsymbol{V}_2$ is purely locally (in material coordinates). However, in order to obtain $\boldsymbol{V}_1$ we need to solve (1.3) which is nonlocal in nature. It is clear that $\boldsymbol{V}_2=0$ whenever $\boldsymbol{V}_1=0$, which is necessary for the special solution (3.8) to hold.

At $n=3$ we find after some algebra

$$\boldsymbol{V}_1\boldsymbol{V}_0^2+\boldsymbol{V}_0^2\boldsymbol{V}_1+2(\boldsymbol{V}_2\boldsymbol{V}_0+\boldsymbol{V}_0\boldsymbol{V}_2+\boldsymbol{V}_0\boldsymbol{V}_1\boldsymbol{V}_0)=0. \tag{3.15}$$

It is clear that (3.15) holds when we have (3.14) showing that the analysis is consistent.

At $n=4$ we find after tedious algebra,

$$\begin{aligned}24\boldsymbol{V}_4&+11(\boldsymbol{V}_1\boldsymbol{V}_0^3+\boldsymbol{V}_0^3\boldsymbol{V}_1)+10(\boldsymbol{V}_0\boldsymbol{V}_1\boldsymbol{V}_0^2+\boldsymbol{V}_0^2\boldsymbol{V}_0\boldsymbol{V}_1)+22(\boldsymbol{V}_2\boldsymbol{V}_0^2+\boldsymbol{V}_0^2\boldsymbol{V}_2)\\&+23(\boldsymbol{V}_0^2\boldsymbol{V}_1\boldsymbol{V}_0+\boldsymbol{V}_1\boldsymbol{V}_0\boldsymbol{V}_0^2)+3(\boldsymbol{V}_1^2\boldsymbol{V}_0+\boldsymbol{V}_0\boldsymbol{V}_1^2)+44\boldsymbol{V}_0\boldsymbol{V}_2\boldsymbol{V}_0\\&+6(\boldsymbol{V}_3\boldsymbol{V}_0+\boldsymbol{V}_0\boldsymbol{V}_3+\boldsymbol{V}_1\boldsymbol{V}_0\boldsymbol{V}_1)+12(\boldsymbol{V}_1\boldsymbol{V}_2+\boldsymbol{V}_2\boldsymbol{V}_1)=0.\end{aligned} \tag{3.16}$$

At this order we find a local formula for $\boldsymbol{V}_4$ once $(\boldsymbol{V}_0,\boldsymbol{V}_1,\boldsymbol{V}_2,\boldsymbol{V}_3)$ are known. Again, if $\boldsymbol{V}_1=\boldsymbol{V}_2=\boldsymbol{V}_3=0$, we automatically have $\boldsymbol{V}_4=0$; which is consistent with the special solution $\boldsymbol{V}(t)=\boldsymbol{V}(0)$.

As anticipated from the left-hand side of (3.9), it imposes constraints on the even-order Taylor coefficients of $\boldsymbol{V}(t)$. In general, $\boldsymbol{V}_{2n}$ may be expressed locally in terms of $\boldsymbol{V}_0,\ \boldsymbol{V}_1,\ \boldsymbol{V}_2,\ \ldots,\boldsymbol{V}_{2n-1}$. While it seems difficult to deduce the general character (e.g. the regularity issues) of the time evolution of the three-dimensional Euler equations from these constraints, they may be used as a solid check of computations of temporal Taylor coefficients for the three-dimensional Euler equations. Moreover, the identity (3.9) may be used in building a model for the three-dimensional Euler equations.

It should be noted that the time derivatives are taken in the sense of Lagrangian sense. It should also be stressed that (3.14, 3.16) are obtained *without* solving potential problems associated with the pressure. In the usual Taylor expansion method, we must solve potential problems to have $\boldsymbol{V}_n$ expressed in terms of $\boldsymbol{V}_j,\ (0\le j\le n-1)$. In this sense the application of Ricatti theory is successful, at least partially, in alleviating the difficulty associated with nonlocality of the three-dimensional Euler equations.

4. Summary

An exact identity for the Jacobian matrix obtained by applying theory of matrix-valued Riccati equations to the velocity gradient in the three-dimensional Euler equations. The essence of the derivation of (3.4) is that we have removed the pressure Hessian by the property (2) of the Riccati equation and by the 2:1 correspondence between the pressure and the velocity.

In general,the governing equations for the Jacobian matrix in the case of $\mathbb{R}^3$ may be derived as

$$\frac{DJ_{ij}}{Dt} = -\frac{\epsilon_{ilk}}{3} J_{lj}(\boldsymbol{x})\omega_n(0)J_{kn}(\boldsymbol{x})$$

$$-\frac{\epsilon_{imk}}{4\pi} J_{lj}(\boldsymbol{x})\mathrm{PV}\int\left(\frac{\delta_{lm}}{|\boldsymbol{x}-\boldsymbol{y}|^3} - \frac{3(x_l-y_l)(x_m-y_m)}{|\boldsymbol{x}-\boldsymbol{y}|^5}\right)\omega_n(0)J_{kn}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}.$$

It should be noted that in spite of the nonlocal character of the dynamics of the Jacobian matrix, the equations for $\boldsymbol{\Phi}(t)$ takes a neat form (3.10,3.11).

A few comments may be in order. The Jacobian $\boldsymbol{J}$ controls regularity of the Euler equations in terms Cauchy invariant. It may be of interest to consider a BKM like criterion in terms of $\tilde{\boldsymbol{J}}$, Beale *et al.* (1984).

By using time-ordered exponentials (e.g. Johnson & Lapidus (2002)), we may express the Jacobian in terms of the velocity gradient

$$\boldsymbol{J}(t) = \mathrm{T}\exp\left(\int_0^t \boldsymbol{V}(t')dt'\right)$$

$$\equiv \boldsymbol{I} + \sum_{n=1}^{\infty}\int_0^t dt_1\int_0^{t_1} dt_2\dots\int_0^{t_{n-1}} dt_n \boldsymbol{V}(t_1)\boldsymbol{V}(t_2)\dots\boldsymbol{V}(t_n),$$

$$\boldsymbol{J}(t)^{-1} = \tilde{\mathrm{T}}\exp\left(-\int_0^t \boldsymbol{V}(t')dt'\right)$$

$$\equiv \boldsymbol{I} + \sum_{n=1}^{\infty}(-1)^n\int_0^t dt_1\int_0^{t_1} dt_2\dots\int_0^{t_{n-1}} dt_n \boldsymbol{V}(t_1)\boldsymbol{V}(t_2)\dots\boldsymbol{V}(t_n),$$

where T ($\tilde{T}$) denotes a (reverse) time-ordered operator. We also have similar formulas for $\tilde{\boldsymbol{J}}(t) = \tilde{\mathrm{T}}\exp\left(\int_0^t \boldsymbol{V}(t')dt'\right)$ and $\tilde{\boldsymbol{J}}(t)^{-1} = \mathrm{T}\exp\left(-\int_0^t \boldsymbol{V}(t')dt'\right)$. However, it seems difficult to deduce (3.9) by solely working with these expressions.

It is hoped that these results will be used in Taylor series analysis of the Euler equations or in developing models for them.

Note added in proof

After completion of this paper, Andrew Gilbert has kindly pointed out to the author that in order to obtain (3.4) we need the assumption just preceding (3.4). He also suggested that a simple ABC flow may be a good test bed for searching for suitable particle paths.

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